On a mean-field model of interacting neurons

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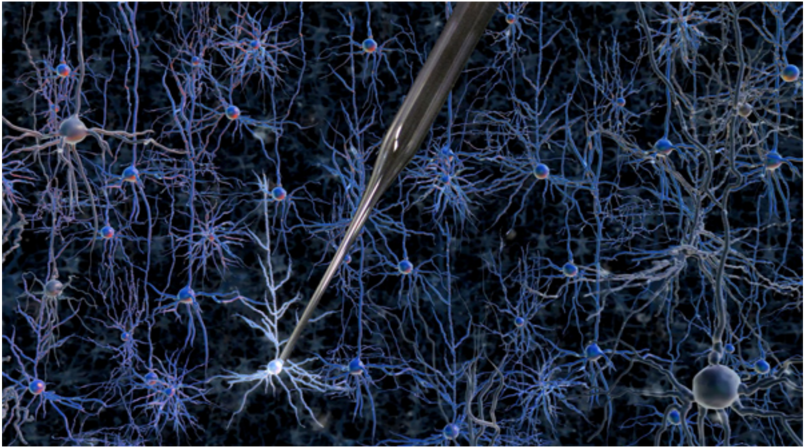
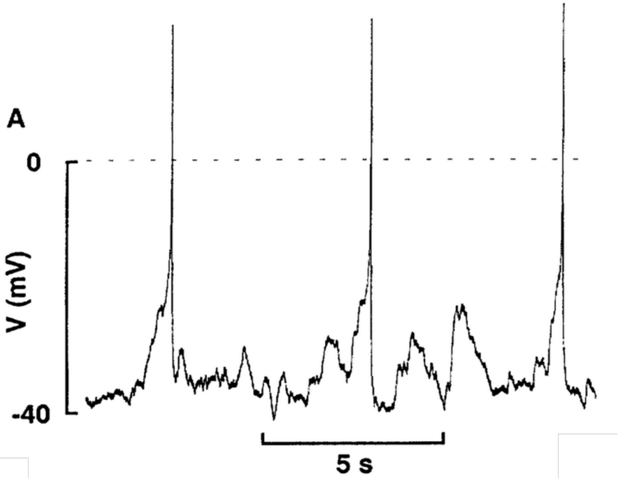
October 22, 2018

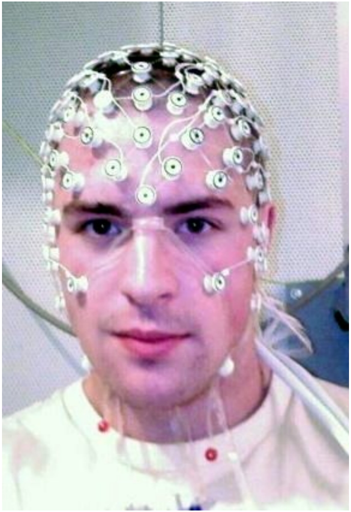
Patch clamping



Patch clamping



EEG



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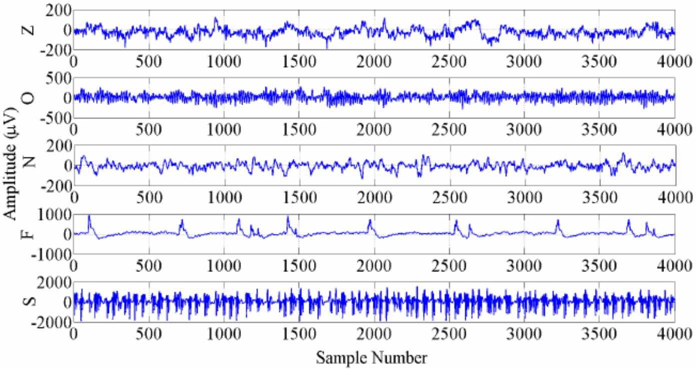

EEG captures the “macroscopic activity” of an assemblies of neurons.

The model (1/2)

N neurons characterized by their membrane potential:

$$V_t^i \in \mathbb{R}_+, \quad t \geq 0, \quad i \in \{1, ..., N\}$$

Between the spikes, $(V_t^i)_{t \geq 0}$ solves a simple deterministic ODE:

$$\frac{dV_t^i}{dt} = b(V_t^i).$$

(Example: $b \equiv$ constant: the potential of each neuron grows linearly between its spikes).

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The model (2/2)

Each neuron i spikes randomly at a rate $f(V_t^i)$.

When such a spike occurs (say at time τ):

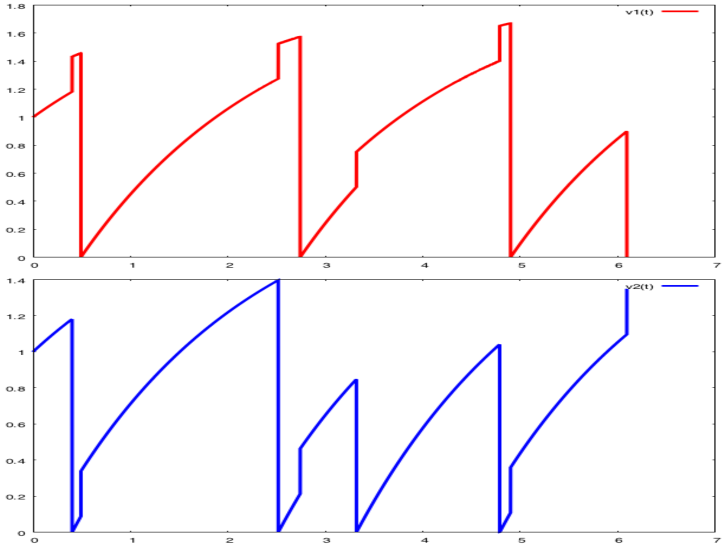
1. The potential of the neuron i is reset to 0:

$$V_\tau^i = 0$$

2. The potentials of the other neurons are increased by J/N:

$$j \neq i,\ V_\tau^j = V_{\tau-}^j + \frac{J}{N}.$$

Illustration with $N = 2$ neurons



The parameters of the problem

The 4 parameters of the model are:

1. the drift $b : \mathbb{R}_+ \to \mathbb{R}_+$, with $b(0) > 0$: it gives the dynamic of the neurons between the spikes
2. the rate function $f : \mathbb{R}_+ \to \mathbb{R}_+$: it encodes the probability for a neuron of a given potential to spike between t and $t + dt$.
3. The connectivity parameter $J \geq 0$.
4. the law of the initial potentials: we assume the neurons are initially i.i.d. with probability law ν.

The parameters of the problem

How should $f \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ behaves?

$$f(x) = (\frac{x}{\vartheta})^{\xi},$$

with: $\xi \geq 1$ (large), $\vartheta > 0$.

When $V_t^i \gg \vartheta$, $f(V_t^i) \gg 1 \implies$ large probability to spike between t and $t + dt$.

When $V_t^i \ll \vartheta$, $f(V_t^i) \ll 1 \implies$ low probability to spike between t and $t + dt$.

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The particle systems

Let $(\mathbf{N}^i(du, dz))_{i=1,\dots,N}$ N independent Poisson measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dudz$.

Let $(V_0^i)_{i=1,\dots,N}$ a family of N random variables on $\mathbb{R}_+$, *i.i.d.* of law ν

Then (V_t^i) is a *càdlàg* process solution of the SDE:

$$\begin{cases} V_t^i = V_0^i + \int_0^t b(V_u^i)du + \frac{J}{N} \sum_{j \neq i} \int_0^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(V_{u-}^j)\}} \mathbf{N}^j(du, dz) \\ \qquad - \int_0^t \int_{\mathbb{R}_+} V_{u-}^i \mathbb{1}_{\{z \leq f(V_{u-}^i)\}} \mathbf{N}^i(du, dz). \end{cases} \tag{1}$$

Propagation of chaos

Let $U_t^j := \int_0^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \le f(V_{u-}^{j,N})\}} \mathbf{N}^j(du, dz)$. The "interaction term" is:

$$J \cdot \frac{1}{N} \sum_{j \neq i} U_t^j$$

Apply (informally !) the law of large numbers:

$$J \cdot \frac{1}{N} \sum_{j \neq i} U_t^j \to_N J \cdot \mathbb{E}\, U_t^1 = J \int_0^t \mathbb{E}\, f(V_u^1) du \text{ as } N \to \infty.$$

The limit equation

We have derived the “mean-field” equation:

$$V_t = V_0 + \int_0^t b(V_u)du + J \int_0^t \mathbb{E}\, f(V_u)du - \int_0^t \int_{\mathbb{R}_+} V_{u-} \mathbb{1}_{\{z \leq f(V_{u-})\}} \mathbf{N}(du, dz)$$

or equivalently:

$$\begin{cases} \frac{d}{dt} V_t = b(V_t) + J\, \mathbb{E}\, f(V_t) \\ + \, (V_t)_{t\geq 0} \text{ jumps to 0 with rate } f(V_t) \end{cases}$$

The non-linear SDE is well-posed.

Theorem

The non-linear SDE has a unique solution, and moreoever:

$$\sup_{t \geq 0} \mathbb{E} f(V_t) < \infty.$$

Propagation of chaos

Theorem (Fournier & Löcherbach 15')

Assume the initial conditions $(V_0^i)_{i\in\{1,\dots,N\}}$ *are i.i.d. with law* ν. *Then* $(V_t^{1,N})_{t\geq 0}$ *goes in law to* $(V_t)_{t\geq 0}$ *as* N *goes to infinity.*

Intuition of the "Propagation of chaos" phenomena:

- Two neurons of the network (say V^1 and V^2) become more and more independent as $N \to \infty$.
- Any neuron of the network (say V^1) looks more and more to the "non-linear" neuron (V_t) as $N \to \infty$.

The Fokker-Planck PDE

The density of V_t (if it exists) is:

$$p(t,x) := \lim_{\Delta \to 0} \frac{1}{\Delta} \mathbb{P}(V_t \in [x, x+\Delta[)$$

Theorem

Assume the initial condition V_0 has a density p_0. Then the law of V_t has also a density $p(t,.)$ and p solves:

$$\begin{cases} \frac{\partial}{\partial t} p(t,x) = -\frac{\partial}{\partial x}[(b(x) + Jr_t)p(t,x)] - f(x)p(t,x) \\ p(t,0) = \frac{r_t}{b(0) + Jr_t}, \quad r_t = \int_0^\infty f(x)p(t,x)dx. \end{cases} \quad (2)$$

Further analysis

What happens for large t ?

- Is the activity of the network going to stabilize ?

$$\mathbb{E}\, f(V_t) \to r \text{ as t } \to \infty \text{ ?}$$

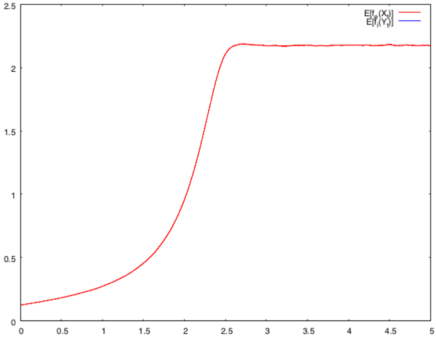
- May spontaneous (stable) oscillations appears?

$$t \mapsto \mathbb{E}\, f(V_t) \text{ tends to oscillate?}$$

Example (movie!)

$$f(x) = x^3, b(x) = 0.4 - x, \; J = 2:$$

$\mathbb{E}\, f(V_t)$



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Relaxation to the equilibrium for small J

Theorem (C., Tanré, Veltz 2018)

Assume the connectivity parameter J is small enough. Then (V_t) has an unique invariant measure which is globally stable: starting from any initial condition, V_t converges in law to the unique invariant measure.

Remarks:

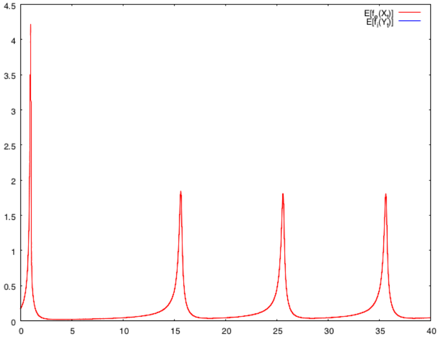
1. No spontaneous oscillations for J small ! The system converges to its unique equilibrium state.
2. Very general result (holds for large class of b and f)
3. The invariant measure has a density, and the density is a stationary solution of the Fokker-Planck PDE:

$$0 = -\frac{d}{dx}[(b(x) + Jr)p(x)] - f(x)p(x).$$

Example (movie!)

$f(x) = x^8$, "$b(x) = -(x - \mathbb{E}\, V_t)$", $J = 1$:

$\mathbb{E}\, f(V_t)$



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Conclusion

- How to go from a finite number of neurons to the mean-field equation $\implies$ propagation of chaos.
- The mean-field equation has a PDE interpretation (the Fokker-Planck equation).
- Small connectivity (J small enough) $\implies$ relaxation to equilibrium.
- Oscillations?

Thank you ! Questions ?