A Convex Approach to the Finite Dimensional Matching Problem In Communication Systems

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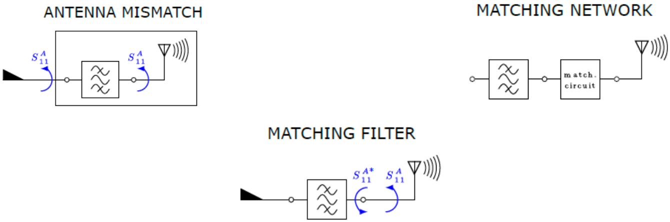
- Continuously increasing interest in wireless networks and connected objects.
- Energy efficient antenna systems providing optimized antenna impedance matching.
- Matching Filter : Co-design of antenna and bandpass filter providing matching and filtering.



Broadband Matching

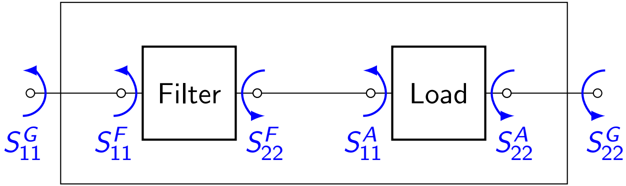


Figure: Global System (Matching Filter and Load)

- Given S_{22}^A, find S_{22}^F which minimize $|S_{22}^G(s)|$ over a given frequency band, $B = \cup_{i=1}^n [a_i, b_i]$.
- Chaining : Rational Schur function S_{22}^F chained with the lossless two port A,

$$S_{22}^G = S_{22}^F \circ A = S_{22}^A(s) + \frac{S_{12}^A(s)S_{21}^A(s)S_{22}^F(s)}{1 - S_{22}^F(s)S_{11}^A(s)} \quad (1)$$

Problem Formulation

- Feasible set, $\mathbb{F}$: set of all rational Schur functions, S_{22}^G such that the load A is de-chainable from S_{22}^G.

$$S_{22}^G \in \mathbb{F} \text{ iff } S_{22}^G(\xi_k) = S_{22}^A(\xi_k),\ \xi_k : \text{simple tr.zeros of load in } \Pi^+.$$

Rational Parametrisation

- Belevitch form of a general loss-less rational scattering matrix,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \frac{1}{q} \begin{bmatrix} \varepsilon p^* & -\varepsilon r^* \\ r & p \end{bmatrix} \quad (2)$$

ε is a unimodular constant, $qq^* = pp^* + rr^*$.

- Let p and r be the reflection and transmission polynomial of G. We have, $r = r^F r^A$, Fix Target Degree of G as $N = deg(r^F r^A)$.
- Let $pp^* = P$ and $rr^* = R$. The modulus square of S_{22}^G,

$$|S_{22}^G|^2 = \frac{pp^*}{qq^*} = \frac{pp^*}{pp^* + rr^*} = \frac{1}{1 + \frac{R}{P}} \stackrel{\text{def}}{=} h(P)^2 \tag{3}$$

- $\mathbb{P}_{2N}^+$ denote the positive polynomials of degree at most $2N$. Define the set,

$$\mathbb{F}_R^N = \left\{ f \in \mathbb{F} \mid \exists P \in \mathbb{P}_{2N}^+ : |f(s)| = h(P) \right\} \tag{4}$$

Matching Problem

Problem ($\mathscr{P}$)

Find $l_{opt} = \min_{S_{22}^G \in \mathbb{F}_R^N} \max_{s \in B} |S_{22}^G(s)|$

Idea : Convex reformulation of Problem ($\mathscr{P}$) using the following notion of admissibiity,

Definition (Admissibility)

For a given load A, the function $h(P)$ is defined to be admissible if there exists a rational Schur function S_{22}^G, satisfying the following,
(i) The load A is de-chainable of S_{22}^G (i.e $S_{22}^G \in \mathbb{F}$).
(ii) $|S_{22}^G(s)| \leq h(P)(s)$.

Convex Formulation

Define the set $\mathbb{H}_R^N$ of positive polynomials as follows:

$$\mathbb{H}_R^N = \{P \in \mathbb{P}_{2N}^+ : h(P) \text{ is admissible}\} \quad (5)$$

Theorem

$\mathbb{H}_R^N$ is a non-empty, closed, convex set of positive polynomials.

Problem $(\mathscr{P})$ can be formulated on the convex set $\mathbb{H}_R^N$ as follows:

Problem $(\mathscr{P}_C)$

Find $\quad L_{opt} = \min_{P \in \mathbb{H}_R^N} \max_{s \in B} \frac{P}{R}(s)$

Solution for Problems ($\mathscr{P}$) and ($\mathscr{P}_C$)

Theorem (Existence and Uniqueness Theorem)

The Problem ($\mathscr{P}_C$) is feasible and there exists a unique $P_{opt} \in \mathbb{H}_R^N$ at which L_{opt} is attained. Moreover, the admissible function, $h(P_{opt})$ will provide a rational Schur function $S_{22}^G \in \mathbb{F}_R^N$ which solves Problem ($\mathscr{P}$).

Theorem (Necessary Condition for Optimality)

When L_{opt} is attained at a $P_{opt} \in \mathbb{H}_R^N$ for Problem $\mathscr{P}_C$, $\max_{s \in B} |S_{22}^G(s)|$ attains $\sqrt{\frac{L_{opt}}{L_{opt}+1}}$ at least $(N+1)$ times.

Characterisation of $\mathbb{H}_R^N$

Admissible set $\rightarrow$ Nevanlinna-Pick Interpolation

$$S_{22}^G(\xi_k) = S_{22}^A(\xi_k), |S_{22}^G(s)| \leq h(P)(s)$$

U_P: Rational Outer Schur function, satisfying, $|U_P(s)| = h(P)(s)$.
Existence of Rational Schur function, $S_P = \frac{S_{22}^G}{U_P}$ satisfying,

$$S_P(\xi_k) = \frac{S_{22}^A(\xi_k)}{U_P(\xi_k)}$$

Proposition

The set $\mathbb{H}_R^N$ defined in (5) is characterised as,

$$\mathbb{H}_R^N = \{P \in \mathbb{P}_{2N}^+ : \Delta(P) \succeq 0\},$$

where Pick matrix, $\Delta(P) = \left[\frac{1-\overline{\left(\frac{S_{22}^A(\xi_i)}{U_P(\xi_i)}\right)}\left(\frac{S_{22}^A(\xi_j)}{U_P(\xi_j)}\right)}{\bar{\xi}_i+\xi_j}\right]_{1\leq i,j\leq m}$.

Optimal Characteristics of Problem $\mathscr{P}_C$

Proposition

For $P_{opt} \in \mathbb{H}_R^N$, at which L_{opt} is attained for problem $\mathscr{P}_C$, the pick matrix, $\Delta(P_{opt})$ is singular.

- $S_{P_{opt}}$ is a blaschke of maximal degree $m-1$.
- Optimal Level for Problem $\mathscr{P}$:

$$S_{22}^{G_{opt}} = S_{P_{opt}} U_{P_{opt}}$$

For degree one antenna - sharp optimal level, higher than degree one antenna - hard bounds for the optimal level attainable.

Solving ($\mathscr{P}_C$) using Non-Linear Semi-Definite Program

Proposition

Let $\mathbb{S}^m$ and $\mathbb{S}^m_-$ denote the set of $m \times m$ Hermitian matrices and Negative Semi-definite Hermitian matrices . The matrix valued function, $\mathscr{A} : \mathbb{P}^+_{2N} \to \mathbb{S}^m$, defined as $\mathscr{A}(P) := -\Delta(P)$, maps $\mathbb{H}^N_R$ to $\mathbb{S}^m_-$, and is convex, i.e $\forall \alpha \in (0,1)$,

$$\alpha \mathscr{A}(P_1) + (1-\alpha)\mathscr{A}(P_2) - \mathscr{A}(\alpha P_1 + (1-\alpha)P_2) \succeq 0$$

$\mathscr{P}_C$: Find $\quad L_{opt} = \min\limits_{P \in \mathbb{H}^N_R} \max\limits_{s \in B} \frac{P}{R}(s)$

- Cost function, $\Psi(P) = \max\limits_{s \in B} \frac{P}{R}(s)$
- $\mathscr{P}_C$: Semi-Definite Programming Problem

$$\min_{P \in \mathbb{P}^+_{2N}} \Psi(P) \qquad (SDP)$$

$$\text{s.t.} \quad \mathscr{A}(P) \preccurlyeq 0$$

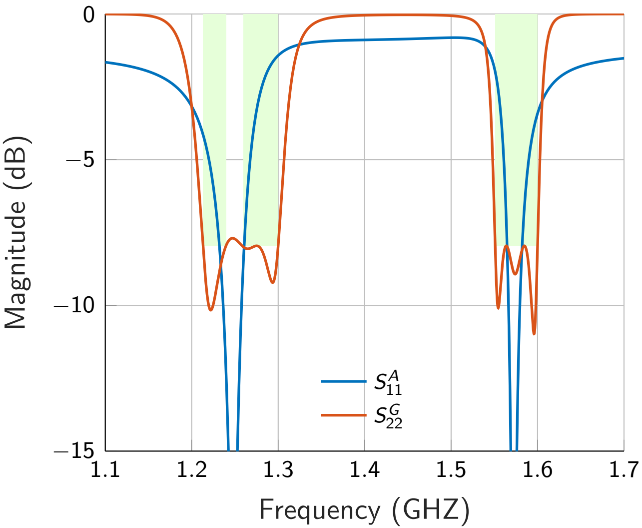


Figure: Result of Problem $\mathscr{P}_C$ (load of degree 2 and system of degree 6).

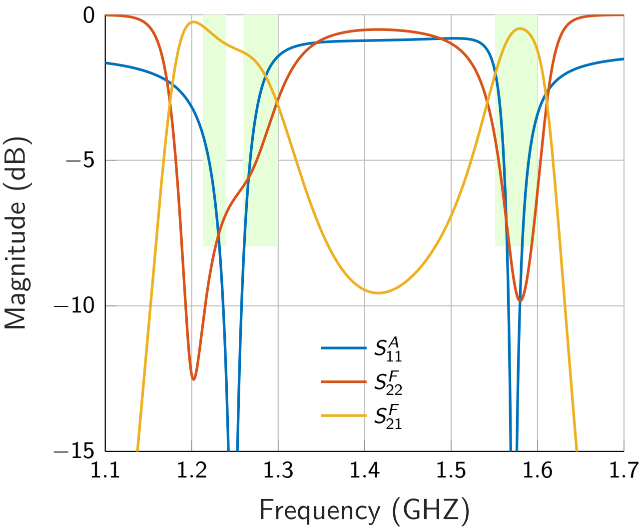


Figure: Matching filter providing the response in previous fig.

Conclusion and Further Work

Conclusion

- The impedance matching problem in communication systems is formulated as a convex optimization problem and the important properties of the problem are discussed.
- Practically feasible method for deriving finite degree matching networks for mismatched antennas.

Further Work

- Critical Point Equation of the Optimisation Problem.
- Building the LC equivalent circuits of Matching Filter and verifying the practical feasibility of proposed scheme.

Thank You